

# Locally conformally Kähler manifolds admitting a holomorphic conformal flow

Liviu Ornea<sup>1</sup> and Misha Verbitsky<sup>2</sup>

## Abstract

A manifold  $M$  is locally conformally Kähler (LCK) if it admits a Kähler covering  $\tilde{M}$  with monodromy acting by holomorphic homotheties. Let  $M$  be an LCK manifold admitting a holomorphic conformal flow of diffeomorphisms, lifted to a non-isometric homothetic flow on  $\tilde{M}$ . We show that  $M$  admits an automorphic potential, and the monodromy group of its conformal weight bundle is  $\mathbb{Z}$ .

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## 1 Introduction

### 1.1 Conformal automorphisms of LCK manifolds

Locally conformally Kähler (LCK) manifolds are, by definition, complex manifolds of  $\dim_{\mathbb{C}} > 1$  admitting a Kähler covering  $\tilde{M}$  with deck transformations acting by holomorphic homotheties. The **monodromy group** of an LCK manifold  $M$  is the deck transform group of the smallest Kähler covering  $\tilde{M} \rightarrow M$ .

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This condition is equivalent to the existence of a global *closed* one-form  $\theta$  (called **the Lee form**) such that the fundamental two-form  $\omega$  satisfies  $d\omega = \theta \wedge \omega$ .

We shall always assume that  $M$  is not globally conformally equivalent to a Kähler manifold.

In the present paper, we prove that any compact LCK manifold which admits a holomorphic conformal flow (*i.e.* an  $\mathbb{R}$ -action by holomorphic transformations of  $\omega$  which are holomorphic with respect to the complex structure  $I$  of  $M$ ) has monodromy  $\mathbb{Z}$ , provided that this flow does not act by isometries on the Kähler covering (Theorem 2.1).

An especially interesting class of LCK manifold is called **LCK manifolds with potential** (see Subsection 1.3). These are manifolds with the Kähler metric on  $\tilde{M}$  admitting a Kähler potential which is automorphic with respect to the action of the monodromy group.

In the present paper, we characterize LCK manifolds with potential in terms of a holomorphic conformal flow. We prove that  $M$  is an LCK manifold with potential if and only if it admits a holomorphic conformal flow which does not act by isometries on the Kähler covering (Theorem 2.3).

## 1.2 Vaisman manifolds

**Definition 1.1:** A **Vaisman manifold** is an LCK manifold  $(M, \omega, \theta)$  with  $\nabla\theta = 0$ , and  $\nabla$  the Levi-Civita connection.

Compact Vaisman manifolds can be characterized in terms of their automorphism group.

**Theorem 1.2:** ([KO]) Let  $(M, \omega)$  be a compact LCK manifold admitting a holomorphic, conformal action of  $\mathbb{C}$  which lifts to an action by non-trivial homotheties on its Kähler covering. Then  $(M, \omega)$  is conformally equivalent to a Vaisman manifold. ■

This characterization is superficially similar to the one given in the present paper for LCK manifolds with the potential (Theorem 2.3). However, we ask for a holomorphic conformal  $S^1$ -action only, and Kamishima-Ornea theorem postulates existence of a holomorphic conformal  $\mathbb{C}$ -action.

Vaisman manifolds are especially important because their topology is easy to control. As shown in [OV2], any Vaisman manifold is diffeomorphic to a locally trivial elliptic fibration over a projective orbifold.

### 1.3 LCK manifolds with potential

Let  $(\tilde{M}, \tilde{\omega})$  be a Kähler covering of an LCK manifold  $M$ , and let  $\Gamma$  be the deck transform group of  $[\tilde{M} : M]$ . Denote by  $\chi : \Gamma \rightarrow \mathbb{R}^{>0}$  the corresponding character of  $\Gamma$ , defined through the scale factor of  $\tilde{\omega}$ :

$$\gamma^* \tilde{\omega} = \chi(\gamma) \tilde{\omega}, \quad \forall \gamma \in \Gamma. \quad (1.1)$$

**Definition 1.3:** A differential form  $\alpha$  on  $\tilde{M}$  is called **automorphic** if  $\gamma^* \alpha = \chi(\gamma) \alpha$ , where  $\chi : \Gamma \rightarrow \mathbb{R}^{>0}$  is the character of  $\Gamma$  defined above.

The Kähler form  $\tilde{\omega}$  on every Kähler covering  $(\tilde{M}, \tilde{\omega})$  of an LCK manifold is by definition automorphic.

**Definition 1.4:** Let  $(\tilde{M}, \tilde{\omega})$  be a Kähler covering of an LCK manifold  $M$ . We say that  $M$  is an **LCK manifold with an automorphic potential** if  $\tilde{\omega} = dd^c \varphi$ , for some automorphic function  $\varphi$  on  $\tilde{M}$ .

As shown *e.g.* in [Ve1], a Vaisman manifold has an automorphic potential, which can be written down explicitly as  $|\pi^* \theta|^2$ , where  $\pi^* \theta$  is the lift of the Lee form to the Kähler covering of  $M$ , and  $|\cdot|$  the metric associated with its Kähler form.

In [OV3], a definition of an **LCK manifold with potential** was given. In this definition, in addition to having an automorphic potential  $\varphi$ , the function  $\varphi : \tilde{M} \rightarrow \mathbb{R}$  was assumed to be **proper**, that is, with compact fibers.

As shown in [OV5, Proposition 1.10], any complex manifold with an LCK metric with automorphic potential admits another LCK metric which also has an automorphic potential,  $\varphi' : \tilde{M} \rightarrow \mathbb{R}$ , but  $\varphi'$  is proper. In the present paper, we prove a stronger result, showing that the monodromy of any LCK manifold with potential is  $\mathbb{Z}$  (Theorem 2.1). This implies that any automorphic potential is proper, and the definition of an LCK manifold with potential in [OV3] is equivalent to Definition 1.4.

We showed in [OV5] that any compact LCK manifold with automorphic potential can be obtained as a deformation of a Vaisman manifold. Many of the known examples of LCK manifolds are Vaisman (see [B] for a complete list of Vaisman compact complex surfaces), but there are also non-Vaisman ones: one of the Inoue surfaces (see [B], [Tr]), its higher-dimensional generalization in [OT], the non-diagonal Hopf manifolds  $(\mathbb{C}^N \setminus \{0\})/\langle A \rangle$  with  $A$

linear, with eigenvalues smaller than 1 in absolute value (see [GO], [OV3]), and the new examples found in [FP] on parabolic and hyperbolic Inoue surfaces.

Compact LCK manifolds with potential are embeddable in Hopf manifolds, see [OV4]. The existence of an automorphic potential leads to important topological restrictions on the fundamental group, see [OV5] and [KK].

The class of compact complex manifolds admitting an LCK metric with potential is stable under small complex deformation ([OV3, Theorem 2.6]). This statement should be considered as an LCK analogue of Kodaira's Kähler stability theorem. The only way (known to us) to construct LCK metrics on some non-Vaisman manifolds, such as the Hopf manifolds not admitting a Vaisman structure, is by deformation, applying the stability of automorphic potential under small deformations.

In [OV6], it was shown that LCK manifolds with automorphic potential can be characterized in terms of existence of a particular subgroup of automorphisms. To state the result, we need to introduce the *weight bundle*  $L \rightarrow M$  associated to the representation  $\mathrm{GL}(2n, \mathbb{R}) \ni A \mapsto |\det A|^{1/n}$ . It is endowed with the flat connection form  $-\frac{1}{2}\theta$ , thus producing a local system. The holonomy of this local system is precisely the monodromy group of  $M$ .

**Theorem 1.5:** ([OV6, Theorem 1.8]) Let  $M$  be a compact complex manifold, equipped with a holomorphic  $S^1$ -action and a LCK metric (not necessarily compatible). Suppose that the weight bundle  $L$ , restricted to a general orbit of this  $S^1$ -action, is non-trivial as a 1-dimensional local system. Then  $M$  admits a LCK metric with an automorphic potential.

**Remark 1.6:** The converse statement is true as well (Theorem 2.3). In the present paper we prove that an LCK manifold  $M$  with an automorphic potential always admits a holomorphic, conformal  $S^1$ -action which lifts to an action by non-trivial homotheties on its covering.

As shown in [OV5, Corollary 1.11], Theorem 1.5 implies the following corollary.

**Corollary 1.7:** Let  $M$  be a compact LCK manifold of complex dimension  $n \geq 3$ . Suppose that the weight bundle  $L$  restricted to a general orbit of this  $S^1$ -action is non-trivial as a 1-dimensional local system. Then  $M$  is

diffeomorphic to a Vaisman manifold, and admits a holomorphic embedding to a Hopf manifold.

## 2 Holomorphic conformal flows on LCK manifolds

### 2.1 Holomorphic conformal flow and monodromy

Let  $M$  be an LCK manifold,  $\tilde{M}$  its minimal Kähler covering, and  $\chi : \Gamma \rightarrow \mathbb{R}^{>0}$  the character defined through the scale factor as in (1.1). Observe first that because we work with the minimal covering, the character  $\chi$  is injective and hence the monodromy  $\Gamma$  can be viewed as a subgroup of the multiplicative group  $\mathbb{R}^{>0}$ . As such, the monodromy group is abelian and torsion-free.

**Theorem 2.1:** Let  $M$  be a compact LCK manifold, and  $\rho : \mathbb{R} \times M \rightarrow M$  a holomorphic conformal flow of diffeomorphisms on  $M$ . Assume that  $\rho$  is lifted to a flow of non-isometric homotheties on the Kähler covering  $\tilde{M}$  of  $M$ . Then the monodromy group of  $M$  is  $\mathbb{Z}$ .

**Proof:** Notice that any conformal holomorphic map  $\varphi : V \rightarrow W$  of Kähler manifolds of dimension  $> 1$  is a homothety. Indeed, the pullback  $\varphi^*\omega_W$  of the Kähler form on  $W$  under a holomorphic morphism is closed. Since  $\varphi^*\omega_W = f\omega_V$ , for some positive function  $f$  on  $V$ , this gives  $df \wedge \omega_V = 0$ , hence  $df = 0$ .

Let  $G$  be the closure of the flow in the Lie group of all holomorphic conformal diffeomorphisms of  $M$ . Then  $G$  is a Lie group. Let  $\tilde{G}$  be its lift to  $\tilde{M}$ , where  $\tilde{M} \rightarrow M$  is the smallest Kähler covering of  $M$ . Then  $\tilde{G}$  is a group of holomorphic homotheties (it was essential to know that the flow is holomorphic to be able to deduce that its lift, formed by conformalities with respect to the Kähler metric, contains in fact only homotheties). Denote by  $\tilde{G}_0$  the subgroup of isometries of  $\tilde{G}$ . We then have the following exact sequence:

$$0 \rightarrow \Gamma \rightarrow \tilde{G} \rightarrow G \rightarrow 0.$$

Since the covering is chosen minimal, the monodromy group does not contain isometries, and hence  $\tilde{G}_0$  injects in  $G$  through  $\tilde{G}$ . We have  $\tilde{G}_0 \hookrightarrow G$ .

But  $\tilde{G}_0$  meets every connected component of  $\tilde{G}$ . Indeed, take an element  $\tilde{a} \in \tilde{G}$  (where  $\tilde{a}$  is a lift of an  $a \in G$ ). It acts on the Kähler form  $\tilde{\omega}$  as  $\tilde{a}^*\tilde{\omega} = C_a \cdot \tilde{\omega}$ ,  $C_a = \text{const}$ . Consider the element  $\rho_{C_a^{-1}}$  of the conformal flow

which satisfies  $\tilde{c}^*\tilde{\omega} = C_a^{-1}\tilde{\omega}$ . Let  $\tilde{b} = \rho_{C_a^{-1}}\tilde{a}$ . Then  $\tilde{b}$  is an isometry with respect to  $\tilde{\omega}$  in the same component with  $\tilde{a}$ . This implies that  $\tilde{G}/\tilde{G}_0 \cong \mathbb{R}^{>0}$ . The natural homomorphism  $\Gamma \rightarrow \mathbb{R}^{>0}$  mapping  $\gamma$  to  $\chi(\gamma)$  (Subsection 1.3) is clearly factorized through the map  $\tilde{G}/\tilde{G}_0 \cong \mathbb{R}^{>0}$ .

To prove Theorem 2.1, we need to show that the image of  $\Gamma$  is discrete in  $\mathbb{R}^{>0}$ . To this end, let  $G_0$  be the subgroup of  $G$  containing all elements which lift to isometries. Obviously, we have an exact sequence:

$$0 \rightarrow \Gamma \rightarrow \tilde{G}/\tilde{G}_0 \cong \mathbb{R} \rightarrow G/G_0 \rightarrow 0. \quad (2.1)$$

The subgroup  $\tilde{G}_0 \subset \tilde{G}$  is a codimension 1 Lie subgroup, which is obvious from the exact sequence

$$1 \rightarrow \tilde{G}_0 \rightarrow \tilde{G} \xrightarrow{\chi} \mathbb{R}^{>0} \rightarrow 0,$$

where  $\chi$  is the scale factor character (1.1), defined in Subsection 1.3. Since  $\tilde{G} \rightarrow G$  is a covering,  $G_0$  is a Lie subgroup of  $G$ . From (2.1) it is clear that to prove  $\Gamma \cong \mathbb{Z}$  it is enough to show that  $G/G_0 \cong S^1$ . As  $\Gamma$  is countable, we find  $\dim(G/G_0) = 1$ . Hence  $G/G_0 = \mathbb{R}/\Gamma$  can be  $\mathbb{R}$  or  $S^1$ . Were it  $\mathbb{R}$ , then  $\Gamma = \{0\}$ , which means that  $M$  is Kähler. It remains  $G/G_0 \cong S^1$  and  $\Gamma \cong \mathbb{Z}$ . The proof is complete. ■

## 2.2 Holomorphic conformal flow and automorphic potential

Applying [OV6], we obtain that a compact LCK manifold which satisfies the assumptions of Theorem 2.1 always admits an automorphic potential. This gives the following corollary:

**Corollary 2.2:** Let  $M$  be a compact LCK manifold, and  $\rho : \mathbb{R} \times M \rightarrow M$  a holomorphic conformal flow of diffeomorphisms on  $M$ . Assume that  $\rho$  is lifted to a flow of non-isometric homotheties on the Kähler covering  $\tilde{M}$  of  $M$ . Then  $M$  admits an automorphic potential, and its monodromy is  $\mathbb{Z}$ .

**Proof** Let  $\omega_0$  be the Gauduchon Hermitian form associated with  $\omega$ . It is a Hermitian form which is conformally equivalent to  $\omega$  and satisfies  $dId(\omega_0^{n-1}) = 0$ , where  $n = \dim_{\mathbb{C}} M$ . As shown in [Ga], such a form always exists, and is unique up to a constant. The constant can be chosen in such a way that  $\int_M \omega_0^n = 1$ , and in this case  $\omega_0$  is preserved by any conformal holomorphic diffeomorphism.

This implies that  $\omega_0$  is  $\rho$ -invariant. Let  $G$  be the closure of  $\rho(\mathbb{R})$  in the group of all holomorphic isometries of  $M$ . Since this group is a compact Lie group (see *e.g.* [MS]),  $G$  is also a compact Lie group, which is obviously commutative, hence isomorphic to a torus. Choosing an appropriate 1-dimensional subgroup in  $G$ , we arrive in the situation described by [OV6]:  $M$  is equipped with a holomorphic action of a circle, which lifts to non-isometric homotheties of its Kähler covering. From [OV6, Theorem 1.8], we now obtain that  $M$  admits an automorphic potential. By Theorem 2.1, the monodromy group of  $M$  is  $\mathbb{Z}$ .  $\blacksquare$

Corollary 2.2 can be used to give a characterization of LCK manifolds with automorphic potential.

**Theorem 2.3:** Let  $(M, I, \omega)$  be a compact LCK manifold. Then the following assertions are equivalent.

- (i)  $M$  admits an automorphic potential.
- (ii) The complex manifold  $(M, I)$  admits an LCK metric  $\omega'$  with same monodromy, and a conformal flow of holomorphic diffeomorphisms of  $(M, I, \omega')$ , which is lifted to a flow of non-isometric homotheties of the Kähler covering  $(\tilde{M}, \tilde{\omega}')$  of  $(M, \omega')$ .

**Proof:** The implication (ii)  $\Rightarrow$  (i) immediately follows from Corollary 2.2. We now show how (i) implies (ii).

Embed  $M$  in a Hopf manifold  $H = (\mathbb{C}^N \setminus \{0\})/\langle A \rangle$ , where  $A$  is a linear (not necessarily diagonal) operator with eigenvalues strictly smaller than 1 in absolute value. Such an embedding was constructed in [OV3]. Then  $A$  preserves the Kähler covering  $\tilde{M}$  and hence can be considered as an element of the deck group  $\Gamma$ . As such it acts as a homothety on the Kähler metric and we can suppose it is a contraction (otherwise we work with  $A^{-1}$ ). What we want is to construct (out of  $A$ ) a holomorphic flow preserving  $\tilde{M}$ .

Recall from [OV3] that the metric completion  $\tilde{M}_c$  of  $\tilde{M}$  is obtained by adding only one point  $z$  (here the existence of the global potential is crucial). Then  $A$  acts trivially on  $z$  and we may consider the local ring  $\mathcal{O}_{\tilde{M}_c}$  at  $z$ .

Observe that  $A$  induces an automorphism of the ring  $\mathcal{O}_{\tilde{M}_c}$ , denoted equally by  $A$ . Then one easily sees that the formal logarithm of  $A$ ,  $\log A$ , is a derivation of  $\mathcal{O}_{\tilde{M}_c}$  (this follows, *e.g.*, from [Bo, p. 209]; it is enough to show that formally  $e^{\log A} = 1$ ). This means that  $\log A$  induces a vector field on  $\tilde{M}$  with associated flow  $e^{t \log A}$ . Note that  $\log A$  is a holomorphic object because, as all eigenvalues of  $A$  are smaller than 1 in absolute value, the

corresponding formal series converges. As  $M = \tilde{M}/\langle A \rangle$ , we see that  $e^{t \log A}$  projects on a one-parameter flow on  $M$ . But, as for  $t = 1$  the flow on  $\tilde{M}$  is  $A$  which acts trivially on  $M$ , the orbits of the projected flow are closed, and hence the projected flow corresponds to an  $S^1$ -action on  $M$ . This action is holomorphic because  $A$  acts holomorphically on  $\tilde{M}$ .

Apply now the averaging (on  $S^1$ ) procedure described in [OV6, 2.1] to obtain a new LCK metric  $\omega'$  on  $M$  with respect to which this  $S^1$  acts by holomorphic isometries. We note that the averaging steps performed do not change the cohomology class of the Lee form, and hence the new LCK structure has the same monodromy.

It remains to justify why the lift of this isometric and holomorphic  $S^1$  to  $(\tilde{M}, \tilde{\omega}')$  is by non-trivial homotheties. This is because the lifted flow contains  $A$  which is a contraction with respect to a certain metric and hence cannot be an isometry with respect to any metric. ■

**Remark 2.4:** The averaging construction used in the proof of Theorem 2.3 preserves the class of LCK metrics with potential. This means that  $\omega'$  of Theorem 2.3 (ii) has an automorphic potential if and only if  $\omega$  has one.

**Corollary 2.5:** Let  $M$  be a compact LCK manifold admitting an automorphic potential. Then the monodromy of  $M$  is  $\mathbb{Z}$ .

**Proof:** By Theorem 2.3,  $M$  admits a flow of holomorphic conformal diffeomorphisms, and by Theorem 2.1 the monodromy group of such a manifold is  $\mathbb{Z}$ . ■

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LIVIU ORNEA

UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS,  
14 ACADEMIEI STR., 70109 BUCHAREST, ROMANIA. and  
INSTITUTE OF MATHEMATICS “SIMION STOILOW” OF THE ROMANIAN ACADEMY,  
21, CALEA GRIVITEI STREET 010702-BUCHAREST, ROMANIA  
Liviu.Ornea@imar.ro, lornea@gta.math.unibuc.ro

MISHA VERBITSKY

LABORATORY OF ALGEBRAIC GEOMETRY,  
FACULTY OF MATHEMATICS, NRU HSE,  
7 VAVILOVA STR. MOSCOW, RUSSIA